

SLAC-PUB-KCL-TH-01-26
hep-th/0107209
July 2001

Coset Symmetries in Dimensionally Reduced Bosonic String Theory

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ABSTRACT

We discuss the dimensional reduction of various effective actions, particularly that of the closed Bosonic string and pure gravity, to two and three dimensions. The result for the closed Bosonic string leads to coset symmetries which are in agreement with those recently predicted and argued to be present in a new unreduced formulation of this theory. We also show that part of the Geroch group appears in the unreduced duality symmetric formulation of gravity recently proposed. We conjecture that this formulation can be extended to a non-linear realisation based on a Kac-Moody algebra which we identify. We also briefly discuss the proposed action of Bosonic M-theory.

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1. Introduction

Early in the development of supergravity it was realised that the scalar fields belonged to a non-linear realisation [1]. Two of the most studied cases are the dimensional reduction on a 7-torus of eleven-dimensional supergravity, which leads to a maximal supergravity whose scalars belong to $E_7/\text{SU}(8)$ coset [2] and the IIB supergravity theory, whose scalars belong to the coset $\text{SU}(1,1)/\text{U}(1)$ [3]. The coset construction was extended [4,5] to include the gauge fields of supergravity theories. This method used generators that were inert under Lorentz transformations and, as such, it is difficult to extend this method to include either gravity or the Fermions. However, this construction did include the gauge and scalar fields as well as their duals, and as a consequence the equations of motion for these fields could be expressed as a generalised self-duality condition.

Recently the entire Bosonic sector, including gravity, of the eleven-dimensional and ten-dimensional IIA supergravity theories were formulated as non-linear realisations [6]. Subsequently [7] it has been argued that eleven-dimensional supergravity and the ten-dimensional IIA supergravity are invariant under a large rank eleven Kac-Moody algebra denoted E_{11} . This group includes the symmetries that are found when the theory is compactified on a n -torus. In particular it was conjectured that M-theory is invariant under E_{11} . It was also proposed that the effective action for the closed twenty-six-dimensional Bosonic string possesses a rank twenty-seven symmetry group denoted K_{27} . In order to encode this symmetry it was shown that pure gravity could be reformulated in terms of two fields, which are related by duality transformation, and was also conjectured to possess an enlarged symmetry algebra. These proposals imply a particular set of symmetries when these theories are dimensionally reduced on an n -torus.

In this note we will systematically discuss the dimensional reduction of the effective action of the (non-supersymmetric) twenty-six-dimensional closed Bosonic string, twenty-seven dimensional Bosonic M-theory (proposed in [8,9]) and pure gravity and find the coset structure of the resulting scalar degrees of freedom. In the case of the Bosonic string, reduced to three dimensions, we obtain the same $O(24,24)$ coset structure that was observed in [5] (also see [10] for the compactification to four dimensions). We will also argue that further compactification to two dimensions leads to a new group consistent with K_{27} . Thus our results agree with the symmetry groups predicted in reference [7] and hence provide support for this proposal. For the case of pure gravity we will find the appearance of an enlarged coset symmetry group in the dimensional reduction to three dimensions consistent with the old results of [11, 12]. We will show that at least part of this symmetry is already present in the duality symmetric formulation of gravity proposed in [7] before it is dimensionally reduced. This encourages the hope that this formulation of gravity possess symmetries that relate many of the solutions of general relativity. We will also suppose that the duality symmetric formulation of gravity can be

extended to a non-linear realisation based on a Kac-Moody algebra. We conjecture that this is a rank D Kac-Moody algebra whose Dynkin diagram is given in figure 1:

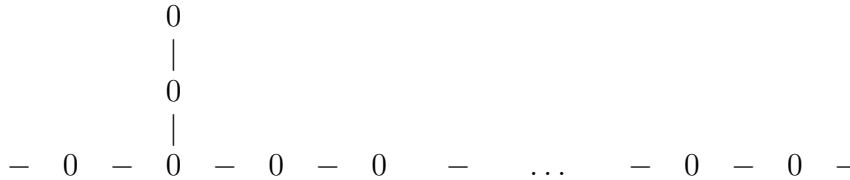


Fig. 1

where the first and last nodes on the bottom line are connected.

The rest of this paper is organised as follows. In section two we discuss the dimensional reduction of various D -dimensional gravitational theories to two and three dimensions. In particular we study the appearance of coset symmetries in the compactified theory. In section three we discuss how these symmetries arise in the uncompactified using the formulation proposed in [7]. Finally we make some comments regarding the effective action of Bosonic M-Theory.

In a sense our discussion in section two is complementary to that of [5], in that we start with a family of higher dimensional actions and ask which ones lead to a coset structure in three dimensions, whereas [5] starts with a coset structure in three-dimensions and then lifts it up to a higher dimensional theory. In addition we will be mainly interested in non-supersymmetric actions. For the wide class of theories we consider, we will see that only a very small set of actions lead to coset symmetries. In particular we will find that M-theory type actions (i.e. gravity coupled to a four-form) only admit a coset structure in the familiar case of eleven dimensions. On the other hand we will find string theory actions (i.e. gravity coupled to a three-form and a dilaton) in all dimensions. However in this case the existence of a coset structure uniquely fixes the coupling of the dilaton to the three-form. From the point of view of dimensional reduction the emergence of a coset structure is somewhat miraculous. However one of the motivations of [7] was to explain these cosets as symmetries in the uncompactified theory. That these coset structures exist beyond the examples of supergravity is further support for the programme of [7].

2. Reducing Bosonic actions

2.1 COMPACTIFICATIONS AND COSET SCALAR LAGRANGIANS

Before starting with the reduction of gravitational theories it will be helpful to first discuss some aspects of compactifications and coset non-linear realisations. We will closely follow the method and construction outlined in [13]. Many of the technical steps required for the dimensional reductions carried out in this section can also be found in [5]. However since the aims of this paper are different, and in particular since we will need to highlight certain points, in this section we give a self contained account of the dimensional reduction procedures and results.

As reviewed in [13], we choose to compactify so as to remain in the ‘Einstein’ frame with a standard kinetic term for the resulting scalar. This fixes the compactification ansatz to be

$$ds_{d+1}^2 = e^{2\alpha_d\phi} ds_d^2 + e^{-4(d-2)\alpha_d\phi} (dx_d + A_\mu dx^\mu)^2, \quad (2.1)$$

where

$$\alpha_d = \sqrt{\frac{1}{2(d-1)(d-2)}}. \quad (2.2)$$

Under such a compactification we find

$$\int d^{d+1}x e R = \int d^d x e \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2(d-1)\alpha_d\phi} F_{\mu\nu} F^{\mu\nu} \right), \quad (2.3)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In addition will also consider m -form field strengths $F_{\mu_1 \dots \mu_m}$. Under the reduction (2.1) we find

$$\int d^{d+1}x \frac{e}{2m!} F_{(m)}^2 = \int d^d x \frac{e}{2m!} e^{-2(m-1)\alpha_d\phi} F_{(m)}^2 + \frac{e}{2m!} e^{2(d-m)\alpha_d\phi} F_{(m-1)}^2. \quad (2.4)$$

Note that it is understood in these equations that the metric quantities are always those of the relevant dimension (e.g. in (2.4) e refers to the $(d+1)$ -dimensional volume element on the left hand side and the d -dimensional volume element on the right hand side).

Clearly we may repeat this procedure to obtain a compactification to on an n -torus. In addition to the n scalars ϕ_i that parameterise the radii of the tori we will also find scalars from the $(m-1)$ -form gauge fields associated to $F_{(m)}$ along the compact directions.

In addition, once we reach $m + 1$ dimensions, we can dualise the m -form field strengths into scalars. To do this we suppose that the action has the form

$$\int d^{m+1}x \frac{e}{2m!} e^{\vec{\alpha} \cdot \vec{\phi}} F_{(m)}^2. \quad (2.5)$$

Next we no longer view $F_{(m)}$ as the curl of a $(m - 1)$ -form but instead introduce a Lagrange multiplier term into the action

$$\int d^{m+1}x \frac{e}{2m!} e^{\vec{\alpha} \cdot \vec{\phi}} F_{(m)}^2 + \frac{e}{m!} \varphi \partial_\mu F_{\nu_1 \dots \nu_m} \epsilon^{\mu \nu_1 \dots \nu_m}. \quad (2.6)$$

Variation with respect to φ simply enforces the Bianchi identity on $F_{(m)}$. However we can also integrate by parts so that there are no derivatives acting on $F_{(m)}$. Eliminating $F_{(m)}$ by its algebraic equation of motion $F^{\nu_1 \dots \nu_m} = e^{-\vec{\alpha} \cdot \vec{\phi}} \epsilon^{\mu \nu_1 \dots \nu_m} \partial_\mu \varphi$ and substituting back into (2.6) leads to the equivalent scalar action

$$\int d^{m+1}x \frac{e}{2} e^{-\vec{\alpha} \cdot \vec{\phi}} \partial_\mu \varphi \partial^\mu \varphi. \quad (2.7)$$

Note the change in sign of the vector $\vec{\alpha}$.

Thus if we compactify all the way down to three dimensions then we can dualise all fields into scalars and put the Lagrangian in the form

$$\mathcal{L} = e \left[R - \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} \sum_{\vec{\alpha}} e^{\vec{\alpha} \cdot \vec{\phi}} \partial_\mu \chi_{\vec{\alpha}} \partial^\mu \chi_{\vec{\alpha}} + \dots \right]. \quad (2.8)$$

Here $\vec{\phi} = (\phi_1, \dots, \phi_n)$, the $\vec{\alpha}$ are constant n -vectors and $\chi_{\vec{\alpha}}$ are additional scalar fields labelled by $\vec{\alpha}$. We see that there are effectively two type of scalars, those labelled by $\vec{\phi}$ and those labelled by $\chi_{\vec{\alpha}}$. The distinguishing feature between these two types of scalars is that various gauge symmetries imply that the $\chi_{\vec{\alpha}}$ scalars can only have derivative interactions. By definition the $\vec{\phi}$ scalars have non-derivative interactions and are obtained from the diagonal components of the metric and we will see that they may be associated to the Cartan subalgebra when a coset structure exists.

In fact our discussion and (2.8) are somewhat over simplified since the off-diagonal metric components (i.e. components of the graviphotons) affect the dualisation argument by altering the Bianchi identity for $F_{(m)}$ in (2.6). In addition

there maybe Chern-Simons terms in the uncompactified theory which will also alter the constraint in (2.6). The effect of these modifications is to replace the kinetic terms $\partial_\mu \chi_{\vec{\alpha}}$ by $A_{\vec{\alpha}}^{\vec{\beta}} \partial_\mu \chi_{\vec{\beta}}$, where $A_{\vec{\alpha}}^{\vec{\beta}}$ is a non-degenerate matrix depending on the scalars $\chi_{\vec{\alpha}}$. However we may safely ignore these subtleties here. Their presence is denoted by the ellipsis in (2.8).

On the other hand we wish to identify (2.8) with a G/H non-linear realisation. To this end we consider a group G with generators \vec{H} and $E^{\vec{\alpha}}$ where the Cartan subalgebra is generated by \vec{H} and

$$[\vec{H}, E^{\vec{\alpha}}] = \vec{\alpha} E^{\vec{\alpha}} . \quad (2.9)$$

We remind the reader that we may split the roots into positive and negative ones. In what follows we call a root positive (negative) if it's first non-vanishing element, as counted from the right, is positive (negative). In addition the positive roots can be written as linear combinations, with non-negative integer coefficients, of the so-called simple roots. The Cartan matrix is defined in terms of the simple roots to be

$$C_{ij} = 2 \frac{\vec{\alpha}_i \cdot \vec{\alpha}_j}{\vec{\alpha}_i \cdot \vec{\alpha}_i} . \quad (2.10)$$

Provided that the roots obey the Serre relation, the form of the Cartan matrix uniquely determines the algebra of G . It will be important later to recall that a defining property of a Kac-Moody algebra is that the off-diagonal entries of the Cartan matrix are negative integers or zero. Every Kac-Moody algebra admits a so-called Cartan involution $\tau : (E^{\vec{\alpha}}, E^{-\vec{\alpha}}, \vec{H}) \rightarrow -(E^{-\vec{\alpha}}, E^{\vec{\alpha}}, \vec{H})$. The Cartan involution can be used to define a subgroup, namely, the subgroup which is invariant under it.

For simplicity we consider non-linear realisations for which the local subalgebras of the Kac-Moody algebra are chosen to be those which are invariant under the Cartan involution. In the case that the subgroup is maximally noncompact the coset representatives can be written as

$$\mathcal{U} = e^{\frac{1}{2} \vec{\phi} \cdot \vec{H}} \sum_{\vec{\alpha} > 0} e^{\chi_{\vec{\alpha}} \cdot E^{\vec{\alpha}}} , \quad (2.11)$$

where the sum is only over the positive roots. It is not hard to show that the scalar

Lagrangian

$$\mathcal{L}_{G/H} = \frac{1}{4} \text{Tr} \left(\partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M} \right) , \quad (2.12)$$

where

$$\mathcal{M} = \mathcal{U}^\# \mathcal{U} , \quad (2.13)$$

is invariant under global G transformations and local H transformations, $\mathcal{V} \rightarrow h(x)\mathcal{V}g$. Here we have used a generalised transpose acting on the generators: $X^\# = \tau(X^{-1})$. In the simplest cases, corresponding to orthogonal subgroups H , $\#$ coincides with the transpose. If we normalise the Cartan and positive root generators so that

$$\text{Tr}(H_i H_j) = 2\delta_{ij} , \quad \text{Tr}(E^{\vec{\alpha}} E^{\vec{\beta}}) = 0 , \quad \text{Tr} \left(E^{\vec{\alpha}} E^{-\vec{\beta}} \right) = \delta^{\vec{\alpha}\vec{\beta}} , \quad (2.14)$$

then one can show that $\mathcal{L}_{G/H}$ is precisely the scalar part of (2.8). Thus it follows that if the vectors $\vec{\alpha}$ obtained from the compactification can be identified with the positive roots of a group G , then the action when dimensionally reduced to three dimensions has a G/H symmetry.

2.2 PURE GRAVITY

It is instructive to first consider the example of pure gravity in D spacetime dimensions:

$$S = \int d^D x e R . \quad (2.15)$$

One class of scalars arises by starting with the curvature scalar in $D-k$ dimensions, then reducing to a 2-form field strength in $D-k-1$ dimensions and then reducing this to a scalar in $D-k-2$ dimensions. In a sense this is the ‘fastest’ way to obtain a χ -scalar starting with the metric in $D-k$ dimensions.

Reducing pure gravity in D dimensions on an n -torus results in $n-2$ such scalars with the roots

$$\vec{\alpha}_k = (0, \dots, 0, -2(D-k-2)\alpha_{D-k-1}, 2(D-k-4)\alpha_{D-k-2}, 0, \dots, 0) , \quad (2.16)$$

where α_d is given in (2.2), $k = 0, \dots, n-2$ and there are k zeros on the left and

$n - 2 - k$ zeros on the right. It is not hard to see that these satisfy

$$\vec{\alpha}_k \cdot \vec{\alpha}_l = \begin{cases} 4 & k = l \\ -2 & |k - l| = 1 \\ 0 & |k - l| \geq 2 \end{cases} \quad (2.17)$$

Other methods of obtaining χ -scalars, i.e. by not going directly from a two-form to a scalar, result in an additional $\frac{1}{2}(n - 1)(n - 2)$ scalars. However their associated roots are not simple but instead take the form $\vec{\alpha}_k + \vec{\alpha}_{k+1} + \dots + \vec{\alpha}_{k+l}$ where $k = 0, \dots, n - 2$ and $l = 1, \dots, n - k - 2$. These roots are summarised in the Dynkin diagram A_{n-1} associated to the group $SL(n)$

$$0 \quad - \quad 0 \quad - \quad \dots \quad - \quad 0 \quad - \quad 0$$

Fig. 2

for $n \leq D - 4$.

Now in three dimensions (i.e. for $n = D - 3$) we can use (2.7) to dualise the graviphotons into scalars. Dualising the first vector that arises (i.e. after compactification to $(D - 1)$ dimensions on a single S^1) leads to the root

$$\vec{\delta} = (2(D - 2)\alpha_{D-1}, 2\alpha_{D-2}, 2\alpha_{D-3}, \dots, 2\alpha_3) . \quad (2.18)$$

This root is positive and, since its first entry is positive whereas none of the $\vec{\alpha}_k$ have a positive entry in the first column, it is also simple. One can show that

$$\vec{\delta} \cdot \vec{\alpha}_k = \begin{cases} 0 & k \neq 0 \\ -2 & k = 0 \end{cases} , \quad \vec{\delta} \cdot \vec{\delta} = 4 . \quad (2.19)$$

On the other hand dualising the remaining vectors in three-dimensions leads to $D - 4$ non-simple positive roots of the form $\vec{\delta} + \vec{\alpha}_0 + \vec{\alpha}_1 + \dots + \vec{\alpha}_l$, $l = 0, \dots, D - 3$. Thus find that the action of gravity reduced to three-dimensions has a coset structure and the associated Dynkin diagram A_{D-3}

$$0 \quad - \quad 0 \quad - \quad 0 \quad - \quad \dots \quad - \quad 0 \quad - \quad 0 \quad - \quad 0$$

Fig. 3

corresponding to the group $SL(D - 2)$. Note that two simple roots, the first and last in the diagram, appear upon compactification to three-dimensions. Thus the Cartan subalgebra is $D - 3$ -dimensional and there are a total of $\frac{1}{2}(D - 2)(D - 3)$ positive roots, with $D - 3$ simple roots.

2.3 GRAVITY COUPLED TO A THREE-FORM

Next we wish to consider the compactifications of actions which include a dilaton ϕ and a three-form field strength $H_{\mu\nu\rho}$. In particular we consider actions of the form

$$S = \int d^D x e \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{\beta \phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right), \quad (2.20)$$

where β is a constant. For a special choice of $\beta = \sqrt{8/D-2}$, (2.20) is the effective action of the Bosonic string (with the tachyon set to zero) or the NS-NS sector of the superstring. The reduction of (2.20) was previously discussed in [10]. However the possibility of dualising the vectors in three dimensions was not performed and hence the corresponding coset was reduced.

Clearly if we dimensionally reduce this system on an n -torus we find the same scalars and roots that we did for pure gravity, coming from the eR term in (2.20). For this action we also begin with an additional scalar ϕ already in D dimensions. The existence of ϕ implies that the putative root vectors are now $(n+1)$ -dimensional. In particular the roots $\vec{\alpha}_k$ discussed in section 2.2 gain an extra column on their left containing a zero. We will also obtain additional putative roots from dimensional reduction of the three-form.

Let us now consider the scalars and their roots that originate from the three-form. The first scalar from $H_{\mu\nu\rho}$ arises when $n=2$ this gives a root

$$\vec{\beta} = (\beta, 2(D-4)\alpha_{D-1}, 2(D-4)\alpha_{D-2}, 0, \dots, 0) \quad (2.21)$$

where there are $n-2$ zeros on the right. A little work shows that

$$\vec{\beta} \cdot \vec{\alpha}_k = \begin{cases} 0 & k \neq 1 \\ -2 & k = 1 \end{cases}, \quad \vec{\beta} \cdot \vec{\beta} = \beta^2 + 4 \frac{D-4}{D-2}. \quad (2.22)$$

Next we need to ensure that the Cartan matrix elements (2.10) are non-positive integers. For $D > 6$ the only possibility is to fix $\vec{\beta} \cdot \vec{\beta} = 4$, i.e. $\vec{\beta}$ is the same length as the $\vec{\alpha}_k$ roots. This in turn determines the coupling to the three-form to be

$$\beta = \sqrt{\frac{8}{D-2}}. \quad (2.23)$$

This dilaton coupling β is precisely that found in the Bosonic string effective action in D dimensions [14].

Continuing with the compactification on an n -torus, with $n \leq D - 5$, we find additional scalars and their (positive) roots. However they are not simple but rather of the form $\vec{\beta} + \sum_k n_k \vec{\alpha}_k$ where n_k are positive integers. These roots are summarised by the Dynkin diagram D_n

$$\begin{array}{ccccccccccc} & & & & & & & & 0 & & \\ & & & & & & & & | & & \\ 0 & - & 0 & - & \dots & - & 0 & - & 0 & - & 0 \end{array}$$

Fig. 4

whose maximally non-compact form is $O(n, n)$.

Upon compactification to four dimensions we may dualise $H_{\mu\nu\lambda}$ to obtain a scalar and its associated root

$$\vec{\gamma} = (-\beta, 4\alpha_{D-1}, 4\alpha_{D-2}, \dots, 4\alpha_4) , \quad (2.24)$$

One can explicitly check that $\vec{\gamma}$ is simple and satisfies

$$\vec{\gamma} \cdot \vec{\alpha}_k = 0 , \quad \vec{\gamma} \cdot \vec{\beta} = 0 , \quad \vec{\gamma} \cdot \vec{\gamma} = \beta^2 + 4 \frac{D-4}{D-2} , \quad (2.25)$$

where $k = 0, \dots, D-6$. Note that $\vec{\gamma}$ automatically has the same length as $\vec{\beta}$.

Continuing to three dimensions we obtain more putative roots by dualising the vectors. However, in contrast to section 2.2 we find that none of these give simple roots. In particular we now find that $\vec{\delta} = \vec{\beta} + \vec{\gamma} + \vec{\alpha}_0 + \dots + \vec{\alpha}_{D-5}$ and hence $\vec{\delta}$ is no longer simple. In addition we find that $\vec{\gamma} \cdot \vec{\alpha}_{D-5} = -2$. Thus the simple roots are those of the Dynkin diagram D_{D-2} and corresponding maximally non-compact group $O(D-2, D-2)$

$$\begin{array}{ccccccccccccccc} 0 & & & & & & & & & & & & 0 & & \\ | & & & & & & & & & & & & | & & \\ 0 & - & 0 & - & 0 & - & \dots & - & 0 & - & 0 & - & 0 & - & 0 \end{array}$$

Fig. 5

Here the roots on the bottom row are the roots $\vec{\alpha}_k$ from section 2.2 and appear from right to left as we compactify, with the first (right most) simple root arising in $D-2$ dimensions and the last (left most) simple root arising in three dimensions. In the top row the right root $\vec{\beta}$ appears after compactification to $D-2$ dimensions and the left root $\vec{\gamma}$ appears upon compactification to four dimensions.

We conclude this section by noting that for $D \leq 6$ there is another possible choice for β that leads to an integral Cartan matrix. Namely we could set $\beta = \sqrt{12 - 2D/D - 2}$ so that the length of $\vec{\beta}$ and $\vec{\gamma}$ is $\sqrt{2}$. However in this case we find that $\vec{\beta} \cdot \vec{\gamma} = 2$, which is positive and hence can not be identified with a Kac-Moody algebra.

2.4 GRAVITY COUPLED TO A FOUR-FORM

Finally we can consider gravity coupled to a four-form $G_{\mu\nu\lambda\rho}$ with no scalar in D dimensions. This occurs in M-theory in eleven dimensions and more recently it was conjectured to be the effective action for a Bosonic M-theory in twenty-seven dimensions [9]. In this case we start with the action

$$S = \int d^D x e \left(R - \frac{1}{48} G_{\mu\nu\lambda\rho} G^{\mu\nu\lambda\rho} \right). \quad (2.26)$$

Clearly when we dimensionally reduce this action on an n -torus we obtain all of the roots we found in section 2.2. In addition we find scalars and their roots from compactifying the four-form. The first such root arises for $n = 3$ and is

$$\vec{\epsilon} = (2(D-5)\alpha_{D-1}, 2(D-5)\alpha_{D-2}, 2(D-5)\alpha_{D-3}, 0, \dots, 0), \quad (2.27)$$

where there are $n - 3$ zeros on the right. It is straightforward to check that

$$\vec{\epsilon} \cdot \vec{\alpha}_k = \begin{cases} 0 & k \neq 2 \\ -2 & k = 2 \end{cases}, \quad \vec{\epsilon} \cdot \vec{\epsilon} = 6 \frac{D-5}{D-2}. \quad (2.28)$$

Further compactification to lower dimensions introduces additional scalars and their roots. However one can check that none of these are simple.

In addition, in five dimensions, we can dualise $G_{\mu\nu\lambda\rho}$ to obtain a new root $\vec{\eta}$, with the same length as $\vec{\epsilon}$. While $\vec{\eta}$ is always a positive linear combination of $\vec{\epsilon}$ and $\vec{\alpha}_k$, the coefficients are integers only if $D = 3a + 5$ for a non-negative integer a . Similarly in three-dimensions $\vec{\delta}$ is a positive linear combination of the other roots but only with integer coefficients if $D = 3b + 2$ for a non-negative integer b .

Thus only eleven-dimensional M-theory has $\vec{\epsilon} \cdot \vec{\epsilon} = 4$ and we find the Dynkin diagram E_8 (as first observed in [1])

$$\begin{array}{cccccccc} & & & & & & 0 & \\ & & & & & & | & \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \end{array}$$

Fig. 6

As above the bottom row of roots $\vec{\alpha}_k$ are those of section 2.2 and appear from right to left as we compactify, starting with the reduction to nine dimensions on

the right and ending in three dimensions on the left. The root $\vec{\epsilon}$ on the top row appears in eight dimensions. For any other dimension the Cartan metric (2.10) does not have integer entries. Hence only eleven-dimensional M-theory possesses a coset symmetry.

We may also consider the effective action of the type II and type 0 strings. Of course it is well-known that the effective action of type IIA and type IIB string theories, when reduced on an n -torus, is identical to the effective action of M-theory reduced on an $n + 1$ -torus. Therefore these theories also have the same coset symmetry as eleven-dimensional M-theory. The type 0A and type 0B string theories differ from the type IIA and type IIB theories in that they do not have any Fermions but instead have double the number of Ramond-Ramond fields. These theories are identical to each other after compactification on an n -torus but differ from eleven-dimensional M-theory. After compactification to three-dimensions they clearly contain all the roots that one finds from compactification of eleven-dimensional M-theory. However there will also be additional roots from the doubled Ramond-Ramond fields. In fact these will just provide a doubling of some roots, but not all, in the compactification of M-theory. Therefore, although one can identify a root system, it is not clear that the action can be written as a non-linear realisation.

2.5 FURTHER REDUCTION TO TWO DIMENSIONS

In the previous three sections we discussed the toroidal compactification of pure gravity, string theory and M-theory from D dimensions to three dimensions. This leads to an action of the form (2.8), consisting only of scalars coupled to gravity. However in three dimensions and less gravity has no degrees of freedom. Thus further compactification will not lead to new scalars in the coset Lagrangian. On the other hand the symmetry of the system is increased upon compactification to two dimensions. In particular, as described in [15], the global G symmetry of the coset action (2.8), when reduced to two dimensions, can be enlarged to an affine symmetry. Thus the total symmetry of group is expected to be an affine version of the groups G found above, obtained by adding a single root in two dimensions to the Dynkin diagrams above. It is natural to assume that this new root $\vec{\zeta}$ has length two and is perpendicular to all the other roots except $\vec{\alpha}_{D-5}$ and $\vec{\delta}$ which arise in three-dimensions. For these roots we assume that $\vec{\zeta} \cdot \vec{\alpha}_{D-5} = \vec{\zeta} \cdot \vec{\delta} = -2$. As a check on this assumption we note that the Dynkin diagrams we obtain in two dimensions for pure gravity and eleven-dimensional M-theory are

$$- \quad 0 \quad - \quad 0 \quad - \quad 0 \quad - \quad 0 \quad - \quad \dots \quad - \quad 0 \quad - \quad 0 \quad - \quad 0 \quad -$$

Fig. 7

where the first and last roots are connected, and

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & | & \\
 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0
 \end{array}$$

Fig. 8

respectively. Thus we obtain the Dynkin diagrams of affine $SL(D-2)$ and E_9 for pure gravity and eleven-dimensional M-theory respectively. These diagrams are those of an affine algebra and are consistent with the results of [11, 12] for four dimensional gravity and agree with the results of [16,17] for M theory.

Returning to the effective action of the Bosonic string, which is our main subject here, we find the Dynkin diagram

$$\begin{array}{ccccccccccccccc}
 & & 0 & & & & & & & & & & 0 \\
 & & | & & & & & & & & & & | \\
 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & \dots & - & 0 & - & 0 & - & 0
 \end{array}$$

Fig. 9

As a further check we note that this is indeed an affine algebra.

3. Symmetries of the Uncompactified Theory

3.1 THE BOSONIC STRING

In [7] it was argued that effective action for the closed Bosonic string is invariant under a Kac-Moody algebra of rank 27, denoted by K_{27} and whose Dynkin diagram is

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & & & & & & 0 \\
 & & & & & & | & & & & & & & | \\
 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & \dots & - & 0 & - & 0 & - & 0
 \end{array}$$

Fig. 10

In this section we will show that the symmetries found earlier in the dimensionally reduced effective action of the closed Bosonic string (here we set $D = 26$) are indeed those predicted in [7]. Let us first recall the derivation of the expected symmetry from the K_{27} algebra. We consider the restriction of the K_{27} algebra obtained by taking all the indices on the generators to lie between 26 and $26 - n + 1$. In terms of the Dynkin diagram of K_{27} in figure 10 this is equivalent to keeping only

the first $n - 1$ nodes on the horizontal line of the Dynkin diagram starting from the right, as well as any nodes to which they are attached by vertical lines. The Dynkin diagram so obtained is D_n provided $3 \leq n \leq 22$, D_{24} if $n = 23$ and affine D_{24} if $n = 24$. It is natural to take the appropriate real form to be $O(n, n)$ for $3 \leq n \leq 22$ and $O(24, 24)$ if $n = 23$. The corresponding local subgroups are taken to be those left invariant under the Cartan involution. For these groups, the local subgroups are then $O(n) \times O(n)$ for $3 \leq n \leq 22$ and $O(24) \times O(24)$ if $n = 23$.

The symmetry groups found in the above restriction of the K_{27} algebra are part of the proposed symmetries of the closed Bosonic string in twenty-six dimensions. However, they are also the symmetries that we expect to find in the dimensional reduction on a n -torus. One way to see this is to note that dimensional reduction on a n -torus leads to a $SL(n - 1)$ symmetry resulting from general coordinate transformations that are preserved by the reduction ansatz. This $SL(n - 1)$ symmetry is just the subalgebra of the Dynkin diagram of K_{27} which results from deleting all nodes in the diagram except the first $n - 1$ horizontal nodes. The identification is confirmed by checking that the group action of $SL(n - 1)$ on the coordinates, as derived from the non-linear realisation, agrees with the general coordinate transformations which are preserved in the reduction procedure.

For $n \leq 22$ one expects to find that the scalars belong to the coset $O(n, n)/O(n) \times O(n)$. In fact the reduction of the effective action of the closed Bosonic string was carried out some time ago in [10] for a n -torus with $n \leq 22$ and these authors did find this coset. It was noted in [10] that in three dimensions one had the possibility to dualise the one form gauge fields and this might change the resulting coset symmetry. In section 2.3 above, upon dualising the vectors in three dimensions, we founded that the scalars belonged to the coset $O(24, 24)/O(24) \times O(24)$ in agreement with [5] and as predicted by the K_{27} algebra of the twenty-six-dimensional theory. This provides an important check on the proposed K_{27} symmetry of the closed Bosonic string.

A further check is provided by the reduction to two dimensions. As discussed above the K_{27} symmetry of the effective action of the twenty-six-dimensional Bosonic string predicts that the symmetry that occurs in the reduction is that found by keeping 23 of the horizontal nodes of the Dynkin diagram and any vertical lines attached to these. Examining figure 10 we see that the symmetry is *affine* $O(24, 24)$. This is precisely the symmetry that occurs in the dimensional reduction as explained in section 2.5.

3.2 PURE GRAVITY

Let us now turn to the analogous discussion of pure gravity. In [7] it was found that in order to exhibit the proposed E_{11} invariance of eleven-dimensional supergravity, its gravitational degrees of freedom would have to be described by a formulation which (when generalised to D dimensions) includes a dual field $h_{a_1 \dots a_{D-3}, b}$ as well as the usual $h_a{}^b$ field. It was suggested that this formulation of gravity could be formulated as a non-linear realisation based on an algebra that included the generators $K^a{}_b$ of $SL(n-1)$ and in addition generators $R^{a_1 \dots a_{D-3}, b}$ associated to $h_{a_1 \dots a_{D-3}, b}$. These additional generators are antisymmetric in their $a_1 \dots a_{D-3}$ indices and satisfy $R^{[a_1 \dots a_{D-3}, b]} = 0$. They obey the commutators

$$\begin{aligned} [K^a{}_b, K^c{}_d] &= \delta_b^c K^a{}_d - \delta_d^a K^c{}_b, \\ [K^c{}_d, R^{a_1 \dots a_{D-3}, b}] &= \delta_d^{a_1} R^{ca_2 \dots a_{D-3}, b} + \dots + \delta_d^b R^{a_1 \dots a_{D-3}, c}, \end{aligned} \quad (3.1)$$

where the ellipsis denotes the appropriate antisymmetrisation.

As above we now consider the subalgebra that is found in the dimensional reduction of pure gravity on an n -torus. This subalgebra is found by only considering the generators whose indices lie between D and $D-n+1$. If $n < D-3$ this subalgebra only consists the generators $K^i{}_j$, $i, j = D, \dots, D-n+1$ of $GL(n)$. However if $n = D-3$ then we must also include the generator $R^{i_1 \dots i_{D-3}, j}$. This latter generator may be written as

$$R^{i_1 \dots i_{D-3}, j} = \epsilon^{i_1 \dots i_{D-3}} S^j. \quad (3.2)$$

It is straightforward to show that the elements

$$\begin{aligned} \hat{K}^i{}_j &= K^i{}_j - \frac{1}{(D-3)} \delta_j^i \sum_k K^k{}_k, \\ \hat{K}^2{}_j &= S^j, \\ \hat{D} &= \sum_k K^k{}_k, \end{aligned} \quad (3.3)$$

generate $GL(D-2)$, except for the generator \hat{K}^{j_2} . The absence of this generator can be accounted for by the fact that the algebra given in (3.1) is only a part of the underlying symmetry of this dual formulation of gravity. This is much the same as the way that G_{11} is only part of the proposed E_{11} symmetry of eleven-dimensional supergravity. If one started from the full underlying symmetry one would expect to find the full $GL(D-2)$ algebra. We return to this point shortly.

As discussed in section 2.2 and 2.3 these are indeed the coset symmetries that result when pure gravity is dimensionally reduced. This includes the enlargement of the symmetry in three dimensions due to the dualisation of the vector fields. This symmetry gives rise to an $SL(2, \mathbf{R})$ symmetry discovered long ago [11, 12] in the reduction of four dimensional gravity. As anticipated in [7] we have shown here that the Borel subgroup of this symmetry is part of the symmetry of the new formulation of gravity proposed in [7], even before it is dimensionally reduced. It is known [12, 15, 17, 18, 19] that this $SL(2, R)$ symmetry becomes affinised if one continues the reduction to two dimensions. These results have proved useful in the construction of solutions of Einstein equations as they rotate one solution into another, for a review see [20]. We expect that this affinised symmetry and the symmetry that results from the reduction to one dimension is also part of the new formulation of gravity given in [7]. This suggests that in this dually symmetric formulation one may find that many solutions of Einstein's theory are in fact related by the symmetries of this formulation.

At first sight this may seem to be a paradox since a symmetry in one formulation of a theory is usually a symmetry in an equivalent formulation. However, one usually takes a symmetry to mean a local symmetry and what may appear as a local symmetry when expressed in terms of the metric may not be a local symmetry when expressed in terms of the dual field. By using both variables in the dual formulation of gravity of [7] one can then find a larger set of local symmetries than appear in Einstein's formulation of gravity.

Although the generators K^a_b and $R^{a_1 \dots a_{D-3}, b}$ can be used to construct a non-linear realisation describing gravity, one might hope to find a non-linear realisation based on a Kac-Moody algebra as was proposed for the maximal supergravity theories and the twenty-six dimensional effective action of the Bosonic string. As in these cases [7], one may try to identify the Kac-Moody algebra by finding the simple positive roots and the Cartan subalgebra generators. Among the generators K^a_b and $R^{a_1 \dots a_{D-3}, b}$ we recognise a set of D commuting generators given by K^a_a . The remaining generators, apart from the negative root generators of $SL(D)$ can be found from multiple commutators of the D generators

$$K^a_{a+1}, R^{4 \dots D, D}, \quad (3.4)$$

which we may identify with the positive simple roots of the rank D Kac-Moody algebra we are searching for. As explained in [7] this procedure is not unambiguous without the negative simple roots. However, the ambiguity may be resolved by identifying the Borel subalgebras of some particular subalgebras.

Clearly the $SL(D)$ Borel subalgebra has the simple positive roots

$$E_a = K^a_{a+1}, \quad a = 1, \dots, D-1, \quad (3.5)$$

and the Cartan subalgebra

$$H_a = K^a_a - K^{a+1}_{a+1} , \quad a = 1, \dots, D-1 . \quad (3.6)$$

The other Borel subalgebra we might like to identify is affine $SL(D-2)$ or $A_{D-3}^{(1)}$. However, the generators considered above do not contain all of this Borel subalgebra. Nonetheless, it is tempting to identify its simple roots as

$$E_a , \quad a = 3, \dots, D-1 , \quad (3.7)$$

and

$$E_D = R^{4\dots D,D} . \quad (3.8)$$

The Cartan subalgebra elements are identified as

$$H_a , \quad a = 3, \dots, D-1 , \quad (3.9)$$

and

$$H_D = K^4_4 + \dots + K^{D-1}_{D-1} - \frac{D-6}{D-2} \sum_a K^a_a . \quad (3.10)$$

One can verify that these do indeed satisfy the relation $[H_a, E_b] = A_{ab} E_b$ where A_{ab} is the Cartan matrix of $A_{D-3}^{(1)}$.

If we assume that these two subgroups have been correctly identified then the simple roots of the Kac-Moody algebra underlying the dual formulation of gravity are E_a , $a = 1, \dots, D$ and Cartan subalgebra is H_a , $a = 1, \dots, D$. One then finds that the Cartan matrix resulting from their commutators corresponds to the Dynkin diagram in figure 1.

4. Bosonic M-Theory

Finally we would like to briefly comment on the effective action of “Bosonic M-theory”. It was proposed in [8,9] that the strong coupling limit of the Bosonic string should be described by a twenty-seven-dimensional theory, called Bosonic M-theory. In particular the authors of [9] proposed a twenty-seven-dimensional action of the form (2.26). However we have seen that this action doesn’t have any coset symmetries when it is dimensionally reduced. On the other hand it is supposed to represent a strong coupling limit of the Bosonic string which, as we have seen, does have a large coset symmetry. It was noted in [9] that the compactification of Bosonic M-theory to the Bosonic string does not quite give the correct action, in particular the dilaton kinetic term is off by a factor of $125/121$. This is effectively the same as the obstruction to obtaining a coset symmetry that we encountered in section 2.4. It was argued in [9] that, since there is no supersymmetry, the reduced Bosonic M-Theory action which is valid at strong coupling does not need to agree numerically with the effective action of perturbative Bosonic string theory. Indeed without supersymmetry or some other symmetry one would not expect it to. However from the point of view taken here these coefficients are fixed by the coset symmetry, which includes T-duality, and one expects that the effective action of Bosonic M-theory should have all the symmetries of the Bosonic string, namely K_{27} .

One might try to invoke a different action for Bosonic M-theory. For example we could consider a three-form instead of a four-form. In this case we encounter the same problem since, as we saw in section 2.3, demanding that the root $\vec{\beta}$ has the correct length is related to the coupling of the dilaton. Therefore proceeding in this manner requires that we also add a dilaton. Indeed with this approach we simply end up with the effective action of a twenty-seven dimensional string theory. Alternatively we could add a dilaton into the Bosonic action of [9]. However in this case one finds that there is no value for the coupling β between the dilaton and the four-form which leads to a Cartan matrix with integer entries. Therefore if we assume that a twenty-seven-dimensional Bosonic M-theory exists and reduces to give a coset structure, then we must conclude that it doesn’t have a simple, local low energy effective action of the form we are familiar with.

5. Acknowledgements

This work was supported in part by the two EU networks entitled "On Integrability, Nonperturbative effects, and Symmetry in Quantum Field Theory" (FMRX-CT96-0012) and "Superstrings" (HPRN-CT-2000-00122). It was also supported by the PPARC special grant PPA/G/S/1998/0061 and N.D.L. is supported by a PPARC fellowship. One of the authors (PAW) would like to thank L. Mason and G. Segal for discussions.

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